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On the transverse M5-branes in matrix theory

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It has been a long-standing problem how the transverse M5-branes are described in the matrix-model formulations of M-theory. We consider this problem for M-theory on the maximally supersymmetric pp-wave geometry, which admits transverse spherical M5-branes with zero light-cone energy. By using the localization, we directly analyze the strong coupling region of the corresponding matrix theory called the plane wave matrix model (PWMM). Under the assumption that the $SO(6)$ scalars in PWMM become mutually commuting in the strong coupling region, we show that the eigenvalue density of the $SO(6)$ scalars in PWMM exactly agrees with the shape of the spherical M5-branes in the decoupling limit. This result gives a strong evidence that the transverse M5-branes are indeed contained in the matrix theory and the theory realizes a second quantization of the M-theory.

Introduction: A nonperturbative formulation of M-theory in the light-cone frame is conjectured to be given by the matrix theory [1]. The matrix theory is expected to achieve second quantization of M-theory, in which all fundamental objects in M-theory are described in terms of the internal degrees of freedom of matrices. It has been shown that there exist matrix configurations corresponding to various objects in M-theory such as supergravitons, M2-branes and longitudinal M5-branes [1–4].

On the other hand, the description of transverse M5-branes has not been fully understood. The charge of the transverse M5-branes is known to be absent in the supersymmetry algebra of the matrix theory, and hence it seems to be impossible to construct matrix configurations for transverse M5-branes with nonvanishing charges.

The absence of the M5-brane charge, however, does not prohibit the presence of M5-branes with compact world-volume, which have zero net charge. It should be clarified whether such compact transverse M5-branes are included in the matrix theory.

The plane wave matrix model (PWMM) provides a very nice arena to understand this problem. PWMM is the matrix theory for M-theory on the maximally supersymmetric pp-wave background of the 11-dimensional supergravity [5]. On this background, M-theory admits a stable spherical transverse M5-brane with vanishing light-cone energy. In general, objects with zero light-cone energy in M-theory are mapped to vacuum states in the matrix theory. Thus, in finding the description of the spherical M5-branes, the target is restricted to the vacuum sector of PWMM.

As we will see below, vacua of PWMM are given by fuzzy sphere and are labeled by the partition of N, where N is the matrix size of PWMM. For each vacuum, the corresponding object with vanishing light-cone energy in M-theory was conjectured in [6]. In particular, vacua corresponding to the spherical transverse M5-brane and its multiple generalization were specified. This conjecture was tested for the case of a single M5-brane by comparing the BPS protected mass spectra of PWMM and those of the M5-brane.

In this Letter, we explicitly show that the spherical M5-brane emerges in the strong coupling regime of PWMM as the eigenvalue density of the $SO(6)$ scalars. We apply the localization method [7] to PWMM and reduce the partition function to a simple matrix integral. By evaluating the matrix integral in the strong coupling limit, we find that the eigenvalue density of the $SO(6)$ scalar matrices forms a five-dimensional spherical shell and its radius exactly agrees with that of the spherical M5-brane in the M-theory on the pp-wave background.

Spherical M5-brane on the pp-wave background: We first review the spherical transverse M5-brane on the pp-wave background. The maximally supersymmetric pp-wave solution of 11-dimensional supergravity is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = -2dx^+dx^- + \sum_{A=1}^{9} dx^A dx^A \\ &\quad - \left(\frac{\mu^2}{9}\sum_{i=1}^{3} x^i x^i + \frac{\mu^2}{36}\sum_{a=4}^{9} x^a x^a\right) dx^+ dx^+, \\ F_{123+} &= \mu, \end{aligned} \tag{1}$$

where μ is a constant parameter corresponding to the flux of the three form field. Throughout this Letter, we use the notation such that $\mu,\nu = +,-,1,2,\cdots,9$, $A,B = 1,2,\cdots,9$, $i,j = 1,2,3$ and $a,b = 4,5,\cdots,9$.

We consider a single M5-brane in this background [6]. Let $X^\mu(\sigma)$ be embedding functions of the M5-brane, where $\sigma^\alpha (\alpha = 0,1,\cdots,5)$ are world-volume coordinates on the M5-brane. The bosonic part of the M5-brane action is given by

$$S_{\rm M5} = -T_{\rm M5}\int d^6\sigma\sqrt{-{\rm det}h_{\alpha\beta}} + T_{\rm M5}\int C_6, \qquad (2)$$

where $h_{\alpha\beta}$ is the induced metric $h_{\alpha\beta} = g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu$ and C_6 is the potential for the magnetic flux $dC_6 = *F_4$. $T_{\rm M5}$ is the M5-brane tension, which is written in terms of the 11-dimensional Planck length l_p as $T_{\rm M5} = \frac{1}{(2\pi)^5 l_p^6}$.

By applying the standard procedure of the gauge fixing in the light-cone frame [8], we obtain the light-cone Hamiltonian of the M5-brane as

$$\begin{aligned} H_{\rm M5} = \int d^5\sigma \Bigg[&\frac{V_5}{2p^+}\left(P_A^2 + \frac{T_{\rm M5}^2}{5!}\{X^{A_1},\cdots,X^{A_5}\}^2\right) \\ &+ \frac{p^+}{2V_5}\left(\frac{\mu^2}{9}(X^i)^2 + \frac{\mu^2}{36}(X^a)^2\right) \\ &- \frac{\mu T_{\rm M5}}{6!}\epsilon_{a_1 a_2\cdots a_6} X^{a_1}\{X^{a_2},\cdots,X^{a_6}\}\Bigg], \end{aligned} \qquad (3)$$

where $V_5 = \pi^3$, p^+ is the total light-cone momentum, P^A are the conjugate momenta of the transverse modes X^A and the curly bracket is defined by $\{f_1,\cdots,f_5\} = \epsilon^{a_1\cdots a_5}(\partial_{a_1}f_1)\cdots(\partial_{a_5}f_5)$.

By noticing that the potential term for X^a in (3) can be rewritten as a perfect square, one can easily find the vacuum configuration as

$$P^A = 0, \quad X^i = 0, \quad X^a = r_{\rm M5}x^a, \qquad (4)$$

where x^a are the embedding functions of the unit 5-sphere into R^6 satisfying

$$x^a x^a = 1, \quad \{x^{a_1},\cdots,x^{a_5}\} = \epsilon^{a_1 a_2\cdots a_6}x_{a_6}. \qquad (5)$$

The constant $r_{\rm M5}$ is determined as

$$r_{\rm M5} = \left(\frac{\mu p^+}{6\pi^3 T_{\rm M5}}\right)^{1/4}. \qquad (6)$$

Thus, we find that the zero energy configuration is a spherical M5-brane with the radius given by (6).

Plane wave matrix model: The action of PWMM is obtained by the matrix regularization of a single M2-brane action on the pp-wave background [5], which is given by the 1+2 dimensional analogue of (2). The bosonic part of the action of PWMM is given by

$$\begin{aligned} S = \frac{1}{g^2}\int dt{\rm Tr}\Bigg[&\frac{1}{2}(DY^A)^2 - 2Y_i^2 - \frac{1}{2}Y_a^2 \\ &+ \frac{1}{4}[Y^A,Y^B]^2 - i\epsilon_{ijk}Y^i[Y^j,Y^k]\Bigg]. \end{aligned} \qquad (7)$$

In obtaining this action, we first apply the matrix regularization, where the embedding functions $X^A(\sigma)$ of the M2-brane are mapped to $N\times N$ Hermitian matrices Y^A as

$$X^A(\sigma^0,\sigma^1,\sigma^2) \to \frac{\mu p^+}{12\pi N T_{\rm M2}}Y^A(t), \qquad (8)$$

and Poisson brackets and integrals on the spatial worldvolume are mapped to commutators and traces of matrices, respectively [13]. The complicated factor in (8) is chosen so that the action (7) takes the simple form. In equation (8), the time coordinate t is related to σ^0 by the same rescaling factor. The coupling constant g^2 in (7) is related to the original parameters in the M-theory by

$$g^2 = \frac{T_{\rm M2}^2}{2\pi}\left(\frac{12\pi N}{\mu p^+}\right)^3, \qquad (9)$$

where $T_{\rm M2} = \frac{1}{(2\pi)^2 l_p^3}$ is the tension of the M2-brane. The covariant derivative D in (7) is defined by $DY^A = \partial_t Y^A - i[A,Y^A]$. This is introduced to take into account the constraint of the form $\{P^A,X^A\} = 0$, which arises in the gauge-fixing in the light-cone frame [8]. In the $A=0$ gauge, the Gauss law constraint correctly reproduces this constraint.

Noticing that the potential for Y^i forms a perfect square, one can easily find the vacuum configuration of PWMM as

$$Y^i = 2L^i, \;\; Y^a = 0, \;\; A = 0. \qquad (10)$$

Here, L^i are N-dimensional representation matrices of the $SU(2)$ generators. The representation can be reducible and one can make an irreducible decomposition,

$$L_i = \bigoplus_{s=1}^{\Lambda} L_i^{[n_s]}, \qquad (11)$$

where $L_i^{[n]}$ stand for the generators in the n-dimensional irreducible representation and $\sum_{s=1}^{\Lambda} n_s = N$. Thus, the vacua are labeled by the partition of N, $\{n_s | n_s \geq n_{s+1}, \sum_s n_s = N\}$.

The conjecture on the spherical M5-brane: The vacuum of the form (10) is the fuzzy sphere configuration and hence it has a clear interpretation as a set of spherical M2-branes. Indeed, the M-theory on the pp-wave background allows zero energy spherical M2-branes as well. The commutative limit of the fuzzy sphere (10), where n_s become large, can naturally be identified with those M2-branes in M-theory.

On the other hand, the correspondence between (11) and the spherical M5-brane can be understood by introducing a dual way of looking at the Young tableau of the vacuum. For the vacuum (11), let us consider the Young tableau corresponding to the partition $\{n_s\}$ such that the length of the sth column is given by n_s. Let us denote by m_k the length of the kth row, where k runs from 1 to max$\{n_s\}$. It was conjectured in [6] that when m_k are large, the vacuum corresponds to multiple M5-branes, where the number of M5-branes is given by $N_5 := \max\{n_s\}$ and each M5-brane carries the light-cone momentum proportional to $m_k (k = 1, 2, \cdots, N_5)$.

In what follows, we test this conjecture focusing on the vacua such that the partition is of the form,

$$L_i = L_i^{[N_5]} \otimes 1_{N_2}. \tag{12}$$

N_2 and N_5 satisfy $N_2 N_5 = N$ and correspond to the number of M2- and M5- branes, respectively. In order to describe the M5-brane, we consider the limit,

$$N_2 \to \infty, \quad N_5 \ : \ \text{fixed}. \tag{13}$$

With the above interpretation, this limit corresponds to N_5 M5-branes, each of which carries the light-cone momentum with an equal amount.

In order to isolate the degrees of freedom of the M5-branes in PWMM, the 't Hooft coupling of PWMM should also be sent to infinity in taking the limit (13) [6]. This can be understood as follows. We first rewrite the metric (1) so that the compactified direction x^- is orthogonal to the other directions as

$$ds^2 = -\frac{\mu^2 r^2}{36} d\tilde{x}^+ d\tilde{x}^+ + \frac{36}{\mu^2 r^2} d\tilde{x}^- d\tilde{x}^- + r^2 d\Omega_2^5 + \cdots, \tag{14}$$

where $r^2 = \sqrt{x^a x^a}$ is the radius of the 5-sphere and $\tilde{x}^+ = x^+ - \frac{36}{\mu^2 r^2} x^-$, $\tilde{x}^- = x^-$. The physical compactification radius is then given by $\tilde{R} \sim R/(\mu r)$, where R is the original compactification radius of M-theory. Upon compactification, transverse M5-branes in the M-theory become NS5-branes in the type IIA superstring theory. The world-volume theory of the NS5-branes is known as the little string theory, which has a characteristic scale given by the string tension $\sim l_s^{-2}$. For the spherical NS5-brane with the radius (6), the theory is controlled by the dimensionless combination $r_{\rm M5}^2 / l_s^2$. We keep this ratio finite to obtain an interacting theory on the NS5-branes while we send $r_{\rm M5}$ to infinity to make the bulk gravity decouple. By using (6) and the well known relation $l_s \sim (l_p^3/\tilde{R})^{1/2}$, we find that the decoupling limit of the NS5-brane is given by $p^+ \to \infty$ with $R^4 p^+$ fixed [14]. The M5-brane in 11 dimensions is recovered by further taking $R^4 p^+$ to be large. Finally, by combining this observation with (13), we find that the decoupling limit of the M5-brane is written in terms of the parameters of PWMM as

$$N_2 \to \infty, \quad N_5 \ : \ \text{fixed}, \quad \lambda \to \infty, \quad \frac{\lambda}{N_2} \to 0, \tag{15}$$

where $\lambda = g^2 N_2$ is the 't Hooft coupling of PWMM. Thus, the decoupling limit of the M5-brane corresponds to the strong coupling limit in the 't Hooft limit.

Spherical M5-branes from PWMM: Let us analyze PWMM in the decoupling limit of the M5-brane by using the localization method. We consider the following scalar field:

$$\phi(t) = Y_3(t) + i(Y_8(t)\sin(t) + Y_9(t)\cos(t)). \tag{16}$$

This field preserves 1/4 of the whole supersymmetries in PWMM and any expectation values made of only ϕ can be computed by the localization method [7]. Since we are interested in PWMM around a specific vacuum (12), we impose the boundary conditions such that all the fields take the vacuum configurations at $t \to \pm\infty$. With this boundary condition, the localization computation leads to the following equality [9–11]:

$$\langle \prod_I \mathrm{Tr} f_I(\phi(t_I)) \rangle = \langle \prod_I \mathrm{Tr} f_I(2L_3 + iM) \rangle_{MM}, \tag{17}$$

where f_I are arbitrary smooth functions of ϕ, $2L_3$ is the vacuum configuration for Y_3, and M is an $N \times N$ Hermitian matrix which commutes with all $L_a (a = 1, 2, 3)$. For the vacuum given by (12), M takes the form, $M = \mathbf{1}_{N_5} \otimes \tilde{M}$, where $\tilde{M}$ is an $N_2 \times N_2$ Hermitian matrix. The expectation value $\langle \cdots \rangle$ in the left-hand side of (17) is taken with respect to the original partition function of PWMM around (12), while that in the right-hand side, $\langle \cdots \rangle_{MM}$, is taken with respect to the matrix integral,

$$Z = \int \prod_i dq_i e^{-\frac{2N_5}{g^2} \sum_i q_i^2} \prod_{J=0}^{N_5-1} \prod_{j=1}^{N_2-1} \prod_{i=j+1}^{N_2} \times \frac{\{(2J+2)^2 + (q_i - q_j)^2\}\{(2J)^2 + (q_i - q_j)^2\}}{\{(2J+1)^2 + (q_i - q_j)^2\}^2}. \tag{18}$$

Here, q_i $(i = 1, 2, \cdots, N_2)$ are the eigenvalues of $\tilde{M}$ [15].

In the decoupling limit (15), the saddle point approximation becomes exact in evaluating the matrix integral. We first introduce the eigenvalue density for q_i by $\rho(q) = \frac{1}{N_2} \sum_{i=1}^{N_2} \delta(q - q_i)$, which is normalized as $\int_{-q_m}^{q_m} dq \rho(q) = 1$. Here, we assume that $\rho(q)$ has a finite support $[-q_m, q_m]$. In the large-λ limit, the saddle point equation of the partition function (18) is reduced to

$$\beta = \pi\rho(q) + \frac{2N_5}{\lambda} q^2 - \int_{-q_m}^{q_m} dq' \frac{2N_5}{(2N_5)^2 + (q - q')^2} \rho(q'), \tag{19}$$

where β is the Lagrange multiplier for the normalization of ρ and we used the fact that $q_m/N_5 \gg 1$ in this limit. The solution of (19) is given by

$$\rho(q) = \frac{8^{\frac{3}{4}}}{3\pi\lambda^{\frac{1}{4}}} \left[1 - \frac{q^2}{q_m^2}\right]^{\frac{3}{2}}, \quad q_m = (8\lambda)^{\frac{1}{4}}, \quad \beta = \frac{8^{\frac{1}{2}} N_5}{\lambda^{\frac{1}{2}}}. \tag{20}$$

By using (17) and the solution for the eigenvalue density (20), we can compute any operator made of ϕ. In particular, let us consider the resolvent of ϕ defined by $\mathrm{Tr}(z-\phi)^{-1}$. According to the result of the localization (17), the expectation value of this operator is equal to that of $\mathrm{Tr}(z-2L_3-iM)^{-1}$ in the matrix integral (18). Note that the support of the eigenvalue density of M is much larger than that of L_3 in the decoupling limit. Thus, we find that, with the suitable normalization as in (8), the spectrum of ϕ lies on the imaginary axis in this limit. This fact enables us to ignore the real part of ϕ in the decoupling limit. Thus, by putting $t=0$, we obtain the following equality in the decoupling limit:

$$\frac{1}{N}\langle \mathrm{Tr}(Y^9)^n(0)\rangle = \int_{-q_m}^{q_m} dq\rho(q)q^n. \tag{21}$$

Namely, $\rho(q)$ is identified with the eigenvalue density of Y^a in this limit.

By using the $SO(6)$ symmetry in PWMM, we then introduce the $SO(6)$ symmetric up-lift of the eigenvalue density of a single Y^a obtained above. We define the uplifted density function $\tilde{\rho}$ as a solution to the equations,

$$\int d^6x^a\tilde{\rho}(r)x_9^{2n} = \left(\frac{\mu p^+}{12\pi N T_{\mathrm{M2}}}\right)^{2n}\int_{-q_m}^{q_m} dq\rho(q)q^{2n} \tag{22}$$

for any n. Here, $r=\sqrt{\sum_a x_a^2}$ and $\tilde{\rho}$ is normalized as $\int d^6x^a\tilde{\rho}(r)=1$. Note that $\tilde{\rho}$ depends only on r because of the $SO(6)$ symmetry. The first factor on the right-hand side of (22) just reflects the rescaling (8), so that $\tilde{\rho}(r)$ can be thought of as a density function in the original target space.

Physical interpretation of $\tilde{\rho}$ is as follows. It will be natural to assume that the $SO(6)$ scalar fields become mutually commuting in the strong coupling limit, since one can always rescale the matrices in such a way that the coupling constant appears only in front of the commutators, so that only mutually commuting configurations contribute to the path integral in the strong coupling limit [12]. Under this assumption, $\tilde{\rho}(r)$ can be thought of as the eigenvalue density of the $SO(6)$ scalars.

The unique solution to (22) is given by a spherical shell in R^6 as

$$\tilde{\rho}(r) = \frac{1}{V_5 r_0^5}\delta(r-r_0), \quad r_0 = \left(\frac{\mu p^+}{6\pi^3 N_5 T_{\mathrm{M5}}}\right)^{1/4}. \tag{23}$$

For $N_5=1$, the shape of the density function (23) exactly agrees with the shape of the spherical M5-brane on the pp-wave background. In particular, the radius r_0 agrees with (6). Thus, under the above assumption, this shows that the transverse M5-brane is formed by the eigenvalue density of the $SO(6)$ scalars in PWMM.

For $N_5>1$, r_0 in (23) is interpreted as the radius of the multiple spherical M5-branes. The N_5-dependence of r_0 coincides with the conjectured form in [6] based on an observation on perturbative expansions in PWMM.

Summary: In this Letter, we considered the matrix theoretical description of the spherical transverse M5-branes with vanishing light-cone energy in M-theory on the maximally supersymmetric pp-wave background. Following the proposal in [6], we considered PWMM expanded around the vacuum associated with the M5-branes. We applied the localization to this theory and obtained an eigenvalue integral. We then analyzed and solved the eigenvalue integral in the decoupling limit of the M5-branes, which corresponds to the strong coupling limit of PWMM. Finally, under the assumption that the $SO(6)$ scalar fields become mutually commuting in the strong coupling region, we found that the eigenvalue density of the $SO(6)$ scalar fields in PWMM in this limit forms a 5-dimensional spherical shell and the radius of the spherical shell exactly agrees with that of the M5-brane in M-theory. Thus, we concluded that the M5-brane in M-theory is formed by the eigenvalue density of the $SO(6)$ scalar fields. We also computed the radius of the multiple M5-branes.

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[13] We use the mapping rules in [8].
[14] Note that we are interested in the case where l_p and μ are fixed.
[15] In the derivation of (17), possible instanton corrections are ignored. However, these corrections are indeed negligible in the decoupling limit (15).